THE
MATHEMATICAL GAZETTE.

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LONDON:
GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

VOL. II. JANUARY, 1903. No. 37.

GENERAL THEORY OF VERNIERS.

II.

3. We have seen the connection between the general theory of verniers and the indeterminate equation of the form $pn - qm = r$; the equation $pn + qm = r$ might be illustrated in much the same way. But, taking the case originally considered, we observe that it is the same question as before, from a different point of view, if instead of starting with an interval AB between divisions of two contiguous scales, and asking how far we must go to reach a coincidence, we suppose the zeros of two scales to coincide and ask how far we have to go to reach a certain interval, between divisions of the two. The question only differs in so far as it suggests the simultaneous consideration of more than one such interval. But further if, instead of repeated divisions of each kind, we take a circle whose entire circumference = one of the larger divisions, a say, and from a fixed point A on the circumference measure off continually arcs = to the smaller division b, (figs. 5, 6), the question how far we have to go to reach a certain approximation to the starting point is the same thing in another form, although the arrangement brings to light many new relations, which in their turn throw light on the result. We have the subdivisions of the different a-intervals superposed as it were upon one a-interval, and the circle is thus continually divided into more and more parts until (if a and b are commensurable) exact coincidence with the starting point is reached. But while the distance x of the pth point of division from A the starting point, (supposing p such intervals to involve q circuits), is defined by the same equation $pb \sim qa = x$ as the distance involved in the vernier problem, there are important results, which did not before present themselves,

founded on the consideration of the way in which the circumference is continually split off into smaller portions as the process of division goes on, by points which, though not consecutive in the order in which they are taken, come near together on the circumference of the circle. These results depend on the obvious principle that if the hth point of division comes at a certain distance from the kth, the $\overline{h+l}$th will come at the same distance from the $\overline{k+l}$th, (whatever these numbers may be). In particular no interval can occur between any two points, which has not already occurred between some point and A. So again if, after inserting p points, we reach a point at a distance x from A, the next p points will occur at distances x, respectively, from the first p. Thus when, in our first circuit, we reach a point B, (fig. 5), at an interval c, $(< b)$ from A, (supposing $a=mb+c$), the next m points will occur at the same distance c, respectively,

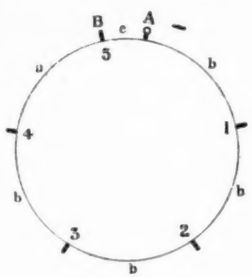


FIG. 5.

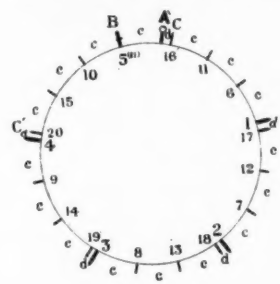


FIG. 6.

from the first m, and so on. Supposing $b=m'c+d$, $(d < c)$, m' circuits will be thus spent in cutting off successive portions c from the b-divisions, until each is divided into m' parts $=c$, and one $=d$. When C' (fig. 6) is thus reached, we have the whole circumference divided into parts of lengths c, d; and the numbers of these parts being $mm'+1$ and m, or p' and p say;

$$a=(mm'+1)c+md=p'c+pd.$$

But the first of these d-segments will be adjacent to A, being that between A and C, where C is the pth point from A. Then and not till then shall we reach a point nearer to A than B was. If q' be the corresponding number of circuits, $(=m'+1)$, $p'b-q'a=d$, (as indeed $qa-pb=c$, where $q=1$).

It is not difficult to see that after the further stage shown in fig. 6 is reached, as described above, a like process will be repeated. In the figure $m=5$, $m'=3$, and, dealing with actual numbers, we have there, at the insertion of the 20th point of division, 16 c-intervals and 5 d-intervals; and the next 16 points

will cut off a length d from each c, and so on, (for the 20th point being at that distance from the 4th the 21st will come at the same distance from the 5th, etc., etc.). Suppose that $c = 4d + e$, $(e < d)$. Then the insertion of 4×16 or 64 points after C', (bringing us up to No. 84), will leave the circumference divided into $64 + 5$, or 69 points each $= d$, with 16 each $= e$. But the first of the e-divisions will occur next to A, (on the side of B), when the 69th point is introduced; while it is obvious that until then we shall never again be so near to A, as we were at C. After No. 84 we shall have a similar cycle of subdivision, ending when the circumference is completely divided into parts equal either to e or to the next smaller division, and so on continually. Segments of three different lengths will occur at intermediate stages.

The general law involved is not difficult to trace. Perhaps the regularity of the process is least obvious, when we come to one of the series m, m'......, which $= 1$. If *e.g.* $m' = 1$, or the division c (introduced with the fifth point in fig. 5), were $> \frac{1}{2}b$, the next smaller division d would be introduced by the insertion of the very next point; but the cycle would be still quite regular, only somewhat compressed, reducing simply to its final stage, corresponding to that from C to C' in the actual case. Using suffixes, the general result may be formulated as follows. After reaching a stage when the circumference is divided into p_r parts each $= a_{r-1}$ and p_{r-1} parts each $= a_r$, (so that we have $p_r a_{r-1} + p_{r-1} a_r = a$), the succeeding points of division will cut off from each of the larger segments in succession a part equal to the smaller, and then from each another like part again, if the larger is more than twice the smaller segment, and so on; and not until we begin the cutting off of the last such part, (which will happen first with the segment next to A), shall we approach so near to A again as we have already been. When the end of this cycle is reached, the circumference will be completely divided into portions of lengths a_r and a_{r+1} $(< a_r)$: it is not difficult to see that the first introduction of such a new smallest segment will be alternately in advance of A and behind it, for successive values of r. Supposing

$$a_{r-1} = m_{r+1} a_r + a_{r+1}, \quad \text{.........................(1)}$$

there will be $m_{r+1} p_r$ points in the complete cycle we are considering. Since each introduces a new division a_r, and we start with p_{r-1} such, the number of these at its close

$$\equiv p_{r+1} = m_{r+1} p_r + p_{r-1}, \quad \text{.........................(2)}$$

so that, with p_r of the smaller divisions left as remainders from the old a_{r-1}-divisions, we have

$$p_{r+1} a_r + p_r a_{r+1} = a. \quad \text{.........................(3)}$$

Moreover the first of the a_{r+1}-segments will be that introduced between the p_{r+1}th point and A, so that we have finally

$$p_{r+1}b-q_{r+1}a=(-1)^{r+1}a_{r+1}, \qquad (4)$$

with a proper value of q_{r+1}, (the sign alternating as pointed out above); and it is not difficult to see that the q's follow the same law of formation as the p's. From the stage defined by (3) we proceed as before. If a, b are commensurable, we shall finally have the entire circumference divided into parts equal to the G.C.M. of a and b.

4. Now these results illustrate graphically, indeed they may be said to amount to a graphical proof of, the fundamental results in the theory of the convergents to a continued fraction. The graphical result we obtained, that after a new shortest division next to A is introduced with the p_rth point, we cannot again approach so near to A until the p_{r+1}th point, as there defined, is reached (with the numerical results which follow), involves the fact that for values of p_r determined in succession by means of equations (1), (2) from obvious initial values, we have $p_rb \sim q_ra=a_r$, a quantity smaller than is possible for any other value than p_r until p_{r+1} is reached; which is the primary result in question.

By means of equation (3) we can deduce other elementary relations. Thus, taking (4) for two consecutive values of r:

$$p_rb-q_ra=(-1)^ra_r,$$

$$p_{r+1}b-q_{r+1}a=(-1)^{r+1}a_{r+1},$$

we deduce

$$(p_rq_{r+1}-p_{r+1}q_r)a=(-1)^r(p_{r+1}a_r+p_ra_{r+1})=(-1)^ra, \text{ by (3)},$$

or

$$p_rq_{r+1}-p_{r+1}q_r=(-1)^r. \qquad (5)$$

Hence again, by means of the same pair of equations,

$$q_{r+1}a_r+q_ra_{r+1}=(-1)^r(p_rq_{r+1}-p_{r+1}q_r)b=b, \qquad (6)$$

a fact which may also be verified graphically. Finally, if (4) be taken for three successive values of the suffixes $r-1$, r, $r+1$, we have by means of (1), (2) the result

$$q_{r+1}=m_{r+1}q_r+q_{r-1}, \qquad (7)$$

although this may also be inferred otherwise. Other points of interest might be mentioned, such as the consideration of the divisions answering to "intermediate convergents," but the above are the results which follow most naturally from the graphical point of view.

PERCY J. HEAWOOD.

DURHAM, *October*, 1902.

GEOMETRY IN FLATLAND.

An inhabitant of Flatland would not be in the same position with regard to elementary geometry as a three-dimensioned being. He could only use superposition for figures of the same aspect. I have not met with any discussion of the way in which he would prove, say, Euc. I. 5; the following seems complicated, but at least it shews that the scope of the usual assumption as to superposition can be made narrower.

The list of theorems that do not depend on the case of superposition in which a figure has to be taken out of its plane includes the following—cases of I. 4 and 26, I. 13-17, 27-30, 32-41, 43, 47, and the theorems of Book II. The following are excluded—cases of I. 4 and 26, I. 5-8, 18-21, 24, 25, 48—we are to prove them by other means.

The constructions of I. 9, 11, 12, 46 are not available, but those of I. 1, 2, 3, 10, 22, 23, 31, 42, 44, 45 may be used. Though we cannot bisect an angle, it is clear that any angle must have a bisector, and only one. The following may be given as a proof:

If the angle BAC has no bisector, any line drawn through A between AB and AC divides it into unequal parts. Suppose AX to be that line, for which $B\hat{A}X$ exceeds $X\hat{A}C$ but by the least possible difference, and AY that for which $Y\hat{A}C$ exceeds $B\hat{A}Y$ but by the least possible difference.

$$\text{Then twice } B\hat{A}X > B\hat{A}X + X\hat{A}C > BAC,$$

$$\text{and twice } B\hat{A}Y < B\hat{A}Y + Y\hat{A}C < BAC,$$

so that

$$B\hat{A}Y < B\hat{A}X.$$

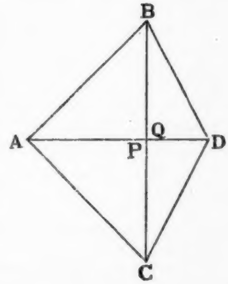
Draw a line AZ between AX and AY, then if $B\hat{A}Z > Z\hat{A}C$ the excess $< B\hat{A}X - X\hat{A}C$, and if $B\hat{A}Z < Z\hat{A}C$ the defect $< Y\hat{A}C - B\hat{A}Y$, which is contrary to the supposition.

Hence the angle BAC must have a bisector.

The fundamental assumption in this proof is that any angle, however small, can be divided into parts.

By taking BAC to be a flat angle we have as a corollary that at any point in any straight line there is a perpendicular.

Through any external point a parallel to this passes. Thus there is one perpendicular (and only one) to any given straight line from any point, whether in it or outside it.



Now let AB be any finite straight line, AD, BC perpendiculars to it at A, B whose lengths are equal to AB. The figure $ABCD$ is a parallelogram (I. 33), and in fact a square. On any straight line there are two squares, one on each side, but these are equal in area (I. 36), and thus the "area of the square on a straight line" is an unambiguous expression. Similarly for a rectangle.

Thus though we may not construct perpendiculars and squares, we may, for the purpose of proving theorems, suppose them drawn, since they certainly exist.

(A) Let ABC be a triangle in which $AB=AC$. Suppose AD to be the bisector of the angle A. Then shall AD bisect BC at right angles.

Let BP, CQ be the perpendiculars from B, C to AD. Cut off $AD=AB$ or AC.

The triangles BAD, DAC are congruent, so that $BD=DC$.

Hence $\quad 2AP\,.\,AD=AB^2+AD^2-BD^2 \quad$ (II. 12 or 13)

$$=AC^2+AD^2-CD^2$$
$$=2AQ\,.\,AD.$$

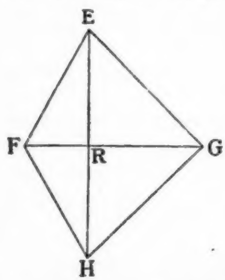
Hence $\quad AP=AQ \quad$ (I. 34 and 40).

Hence $\quad BP=QC \quad$ (I. 47).

Since $AP=AQ$, P and Q coincide, and thus the proposition is proved.

It follows that the perpendicular from A to BC bisects the angle A and the line BC, and that the median from A bisects the angle A and is perpendicular to BC.

(B) Now let EFG, HFG be two triangles on opposite sides of FG, having $EF=HF$ and the angles EFG, GFH equal. Then shall the triangles be equal in all respects.



Draw EH cutting FG, produced if need be, in R. By (A), FG bisects EH at right angles.

Hence $\quad EG=HG \quad$ (I. 47).

Since EGH is isosceles GR bisects the angle EGH. Hence the angles EGF, FGH are equal.

The third angles are equal (I. 32) and the areas (I. 38).

The second (symmetrical) case of I. 4, and I. 5, follow at once from this. I. 5 could also be deduced from (A) and I. 32.

For I. 8 in the symmetrical case put the bases together and use I. 5. In the congruent case compare the triangles with a symmetrical one.

In the symmetrical case of I. 26, which includes I. 6, compare the triangles with a third, made symmetrical with one of them by I. 23 and 3.

A. C. Dixon.

MATHEMATICAL NOTE.

119. [I. 13. b. α; 17. c.]

Rule for finding the number of quarts not greater than a given number N (a quart being a number which cannot be expressed as the sum of one, two, or three squares).

1. Express N in the binary scale, and let its digits, commencing with the unit-digit, be a_0, a_1, a_2, a_3, etc. (Of course each a is either 0 or 1.)

2. In this expression for N note sequences of three 1's; but only count such sequences of three as end with the unit-digit, or with a digit an *even* number of places from the unit-digit. Let β be the number of these sequences.

3. Then the number of quarts not exceeding N will be

$$\beta+(a_3+a_5+a_7+\ldots)+2(a_4+a_6+\ldots)+4(a_5+a_7+\ldots)$$
$$+8(a_6+a_8+\ldots)+\text{ etc.}$$

Examples of the Rule.

Ex. 1. To find the number of quarts not exceeding 125.

$125=1111101$ binary, and there are two sequences as marked. Hence the number of quarts is

$$2+(1+1)+2(1+1)+4\cdot 1+8\cdot 1=20.$$

Ex. 2. To find the number of quarts not exceeding 167.

$167 = 10100\overparen{111}$ binary, and there is one sequence as marked. Hence the number of quarts is

$$1+(0+1+1)+2(0+0)+4(1+1)+8\cdot 0+16\cdot 1=27.$$

N.B.—These 27 quarts are—7, 15, 23, 28, 31, 39, 47, 55, 60, 63, 71, 79, 87, 92, 95, 103, 111, 112, 119, 124, 127, 135, 143, 151, 156, 159, 167.

W. A. WHITWORTH.

REVIEWS.

Cours d'Analyse Mathématique. Tome I. By É. GOURSAT. Pp. 620. 1902. (20fr.)

This work is practically the résumé of a course of lectures given by the author to the Faculté des Sciences de Paris. The attempt has been made to give in this the first volume a general exposition of the properties of functions of real variables, the exception being made of those connected with differential equations. This part of the subject, however, will be treated in a second volume at present in the press.

As was perhaps to be expected from M. Goursat, the book is one of the most pleasing which has appeared recently on the subject in question. The matter is treated in a vigorous manner, and its arrangement leaves little to be desired. The author assumes that the student has some acquaintance with the elements of the calculus, and he also assumes some knowledge of the better known properties of irrationals. Whilst admitting that the theory of irrationals ought logically to form the ground-work of an exposition of Mathematical Analysis, appeal is made to the many well-known works on the subject.

The contents of the volume are divided primarily into four parts—(1) General theorems on differentiation, (2) Integration and properties of Definite Integrals, (3) Theory of Series, and (4) Geometrical Applications.

The first part commences with theorems on continuity and on limits. General theorems connected with differentiation are then given, some space being devoted to a consideration of the properties of Jacobians and of Hessians, and to a discussion of various transformations, such as those of contact, for example.

Taylor's Theorem, and its extension to several variables, together with properties of functions connected with it, form the next chapter, while the remainder of this part is devoted to a discussion of maxima and minima, and to problems connected therewith.

The second part commences with a close examination into the question of continuity from an analytical standpoint. General theorems on integration are then given, which are afterwards exemplified by appeal to geometrical intuition. After a discussion of ordinary methods of integration the author proceeds to consider double and multiple integrals, some space being devoted to Green's and Stokes' theorems. A few interesting examples of definite integrals are given, and this part concludes with a short account of the integration of total differentials.

The next two chapters are concerned with the theory of series. The first of these is occupied with general convergence criteria. For the purpose of obtaining many of them, use is made of Cauchy's theorem that

$$\sum_{x=0}^{\infty} \phi(a+x) \text{ and } \int_a^{\infty} \phi(x)dx$$

converge or diverge together.

There are also given in this part of the work sections on multiple series and on series with variable terms. The section on properties of power series is particularly worthy of note. The division concludes with a discussion of trigonometric series, the proof of Fourier's theorem given being Bonnet's modification of that due to Lejeune-Dirichlet.

The remainder of the volume is occupied with general properties of plane and twisted curves and of surfaces.

An interesting feature of the work as a whole is the number of examples given, both solved and unsolved. Those solved seem particularly adapted to illustrate the points desired, and the exercises given at the ends of the various chapters form a collection which is sure to be extremely useful.

Altogether the author is to be congratulated on having produced a very interesting elementary treatise on Mathematical Analysis, and one which is sure to be found widely useful. J. E. WRIGHT.

Elementary Geometry. W. M. BAKER and A. A. BOURNE. Pp. viii. + 211. 2s. 6d. Second Edition revised. 1902. (Bell.)

The appearance of this and similar books marks an epoch in the history of geometry. The time seems to have come when Euclid, considered as an introduction to geometrical ideas, must at last be abandoned in this country. Even as a system of logic he falls before the searching requirements of the modern founders of geometry, and it would appear that the pre-eminent place he has occupied for over two thousand years must be vacated. It cannot be without a feeling of sentimental regret that we watch him being relegated to antiquity while our better judgment bids us welcome the new dispensation.

The present volume is written on the lines suggested by the Committee of the Mathematical Association (see *Gazette*, May, 1902), and includes the substance of Euclid, Books I., III., 1 to 34, and IV., 1 to 5, theorems on loci, mensuration, and an appendix on graphs, which, though doubtless a useful introduction to the subject, seems rather out of place here. The remainder of the course is to be published shortly.

The recommendations of the Committee have been very faithfully followed, and the result is in essentials excellent, and should prove extremely useful to teachers. The work has been so well done that we are tempted to find fault with details which might otherwise have passed unnoticed. For we do not regard it as impossible to establish some one standard book on geometry in place of Euclid, and are convinced that in the interests of education such a book is very much to be desired. But while being based on sound lines and wide knowledge, it must be free from superficial blemishes if it is to last for all time, and some of these we are constrained to point out in the present volume.

The definition of a straight line has always been a difficulty. To say that it is "even" is only to substitute one adjective for another. It is better not to define, but to state the chief property, namely, the unique determination by two points. So too the idea of an angle is distinct from, and not less simple than, the idea of amount of turning. It, like straightness, is a fundamental intuitive idea, and should not be defined; but the association of rotation with it is extremely valuable, and it is strange, considering the practical nature of the book, that the suggestion of the Committee has not been followed in this case, and that the opportunity of "illustration by rotation" in dealing with the exterior angles of a polygon is missed. A distinction between different directions of rotation should be made, as without it the reasoning on page 3 is invalid.

It seems an unnecessary piece of conservatism to place the equality of right angles among the axioms, especially as a proof is given on the next page.

A list of abbreviations is given on p. 10, which "may be used" (presumably by students) "in writing out propositions." This may be useful in establishing uniformity, but the continual use of unnecessary abbreviations when not required for compactness is disfiguring to the text and annoying to the reader.

The English language has been called a branch of mathematics, and an English text-book should be free from even time-honoured abuses. The word "shall" as used in enunciations is barbarous and unmeaning, and the authors are not even consistent in its use.

A few minor details may be noted. The name pentagon is used on p. 19 and defined on p. 24. To *prove* prop. 4 by cutting triangles out of a *double* sheet of paper is rather begging the question. In the construction of a triangle with given sides, it should be pointed out that the circles must *cut*. It is surprising to come suddenly upon the definition of a chord on p. 171 after frequent use has been made of the term and after the definition has been given in its proper place at the beginning of Book III.

The book is well supplied with examples both of a theoretical and of a practical character. R. W. H. T. HUDSON.

Elementary Geometry. J. ELLIOTT, M.A. 4s. (Swan Sonnenschein & Co.)

This is an attempt to supplant Euclid: his form of proof is retained, but his order is disregarded; angles at a point and parallels come first; the triangle, congruent triangles, and an admirable section on the solution of problems; then inequalities, circles, and lastly areas. This order seems excellent in many ways; but, if we are to have much measurement, it will be convenient to take areas much earlier, and I. 47, which comes right at the end, should be taken at the same time as square root in arithmetic. The explanations of the definitions and of the fundamental methods employed, and the hints about proving theorems and solving problems, are all admirable; in some cases the mathematician may think them too obvious to be worth reading, but to the non-mathematician and less experienced teacher they will prove of great assistance; as for the pupil he will not read them—they are too long. In fact, the whole book is too long and too full for actual school use. Such a book ought to be the teacher's companion to a more concise book to be placed in the hands of the class.

An angle is defined as the difference of two directions, and parallel straight lines as lines in the same direction. The author defends these on the grounds that they have proved successful in class teaching. Even considering the clear illustrations that lead up to the definition of an angle, it is impossible to accept it without a shudder. Would it not have been wiser to give the illustrations, and say that an angle cannot be satisfactorily defined but that certain axioms will be assumed?

It is a pity the author has not shown how experimental work should be interwoven with the formal geometry. Many numerical examples and illustrations might have been given.

The size of the book (there are 268 pages, and the ground covered is only Euclid I.-IV.) might have been decreased with advantage by using the usual abbreviations. In general, it is easier to follow a proof in which they are used. The printing and figures are not all that could be desired. There are several misprints (mostly unimportant, it is true), and in one figure several lines are left out. Why is a figure often repeated on the *opposite* page? A. W. SIDDONS.

A Philosophical Essay on Probabilities. By PIERRE SIMON, MARQUIS DE LAPLACE (translated by F. W. TRUSCOTT and F. L. EMORY). Pp. iv., 196. 2 dols. 1902. (Wiley, New York; Chapman & Hall.)

Probabilités et Moyennes Géometriques. By E. CZUBER (translated into French by H. SCHUERMANS, with a preface by C. LAGRANGE). Pp. xii., 244. 8 fr. 50. 1902. (Hermann, Paris.)

Wahrscheinlichkeitsrechnung, und ihrer Anwendung auf Fehlerausgleichung, Statistik, und Lebensversicherung. By E. CZUBER. Part I. Pp. 304. 1902. (Teubner, Leipzig.)

"The gambling fraternity will continue to proclaim their belief in luck—and the community on whom they prey will, for the most part, continue to submit to the process of plucking, in full belief that they are on their way to fortune." Thus wrote R. A. Proctor, in the preface to his serious and amusing little volume, *Chance and Luck*. A member of the gambling fraternity who purchases any of the above publications with a view to improving his chances on the turf, at Monte Carlo, or in the Continental lotteries, will be as edified as was the too confiding farmer who bought Ruskin *On Sheepfolds*. An angel from heaven could not convince this class of mind that if a penny has turned up tails ten times running, it is still an even chance that it turns up heads or tails on the next throw. The nearest approach to sanity that such a being might conceivably exhibit would be the suggestion that the coin in all probability was loaded. To him, therefore, a philosophical treatise such as that of Laplace will in nowise appeal. The books under notice are for those "who await no gifts from fate," even if it is too soon to say that, as Arnold goes on to add, "they have conquered fate." Unfortunately, the intrinsic nature of the subject is one that may mislead the acutest minds as well as the ordinary layman. As De Moivre pointed out, the reason of the layman's confidence is the deceptive character of the problems of chance, which seem as if they but require ordinary sound sense for their solution. Even when

error has crept into solutions by expert hands, it is often exceedingly difficult to detect the source of the misconception. This will, no doubt, account in a large measure for the fascination which this subject has exercised on piercing intellects, and perhaps that logical divine, Archbishop Whately, had this in his mind when he sat down to compose his ingenious pamphlet: *Historic Doubts Respecting Napoleon Bonaparte.*

Roughly speaking, Pascal and Fermat divide the glory of founding the theory, the views of the latter being, indeed, more accurate, as far as fundamental principles are concerned, than those of his friend and rival. The famous numbers of James Bernouilli make their first appearance in his *Ars Conjectandi.* Cotes "of Trinity," of whom Newton said that if he had lived we should have learned something, was the first to explore the theory of errors. The theory of recurring series saw the light in De Moivre's *Doctrine of Chances.* Thomas Simpson carried on the work of James Dodson (who wrote a century before) in his *Laws of Chance* and *Theory of Annuities and Reversions.* All this and more we may find in the pages of Todhunter's classic treatise on the history of the subject. We may add *en passant* that it is surely time that we had a reprint of this famous volume, which has been out of print for many years, and can only be purchased at a price beyond the reach of the shallow purse.

Just as Laplace published a popular *Système du Monde* as a pendant to his *Mécanique Céleste*, so to his great *Théorie Analytique des Probabilités* he writes for his contemporary public the book which heads our list. The translation under notice cannot be said to be entirely successful. On the whole it is faithful and straightforward enough, but it is uneven, and in many places ungraceful. The table of errata would seem to betray the cause. For "Pline" read "Pliny"; for "sun" read "soil"; for "primary" read "prime." The translators are professors respectively of Germanic languages and of Applied Mathematics. It would be interesting to know whether the Professor of German languages confused *soleil* with *sol*, and did not know of the existence of *prime* numbers, and if Pliny was to him but a shadowy personality. The Teuton Ph.D., who knows the value attached to a "Von," should not have deprived De Moivre of his badge of aristocracy, shadowy though it may have been. In fact, his name is generally written *Demoivre.* Roger Cotes is presented with a circumflex—Côtes—though Leicester born and bred. For Leibnitz we have "Liebnitz, always led by a singular and very loose metaphysics" (!) where Leibniz or Lubeniecz are the only lawful variations. The niece of Pascal, heroine of the miracle of the Holy Thorn, was surely Marguerite Perier and not Mlle. Perrier. Ticho-Brahé looks very strange to English eyes. The few passages in which mathematics is directly introduced are none the easier to read for being expressed, as in the following instance: We "have Z equal to the binomial $T-1$; the product of V by the nth power of Z will thus be equal to the product of V by the development of the nth power of the binomial $T-1$." Surely the translators might have taken the bull by the horns, and, to mingle metaphors, they might have given us in approximately one page this "linked sweetness long drawn out" into $6\frac{1}{2}$ pages.

The best English summary of Laplace's reasoning in his great book on probabilities is given by De Morgan in the *Encyclopaedia Metropolitana.* The manner of using his results, as De Morgan said elsewhere, may be described to a person who has a nodding acquaintance with decimal fractions. We may say that a knowledge of vulgar fractions will almost suffice for the comprehension of the book under notice. It deals with the general principles underlying the doctrines of probability and hope, and their application to games of chance, to astronomy, testimony, the decisions of assemblies and tribunals, and to tables of mortality. Among the more interesting of the allusions to contemporary topics of discussion is that to Pascal's argument, reproduced under a geometrical form by Craig, "the English mathematician." "Witnesses declare that they have it from Divinity, that in conforming to a certain thing one will enjoy, not one or two, but an infinity of human lives. However feeble the probability of the proofs may be, provided that it be not infinitely small, it is clear that the advantage of those who conform to the prescribed thing is infinite, since it is the product of this probability and an infinite good; one ought not then to hesitate to procure for oneself this advantage." Laplace puts his tongue in his cheek and the happy lives in an urn, and with some unction demolishes the argument of Pascal. The book is full of interest even in its present dress. Its perusal is, of course,

infinitely less arduous than the task of mastering the *Théorie Analytique*. As Morgan Crofton, referring to the greater work, concludes his article in the *Encyclopedia Britannica*: "It is, and will long continue to be, one that must be attempted by all who desire to understand and to apply the theory of probability."

The second of the books with which this notice is headed is a translation of Czuber's *Geometrische Wahrscheinlichkeiten und Mittelwerthe*, published at Prague in 1884—a work on local probability which received the honour of special mention in the article to which we have just alluded. It consists of a series of problems—*joyeux devis*—many of which lead to the most fascinating of paradoxes, due in general to the sense in which the solver understands the much disputed term "at random." It is gratifying to find that the author has largely drawn on the hundreds of problems in the *Educational Times* connected with the names of Blackwood, Clarke, M'Coll, Seitz, Sylvester, Watson, Wolstenholme, and Woolhouse. It is also gratifying to find in its right place a name which naturally received but a bare mention in the *Encyclopedia* article, that of Morgan Crofton himself, whose great memoir in the *Phil. Trans.* of 1868 entitled him to be called the father of this branch of investigation. As Herr Czuber justly observes: "Crofton's methods are alike distinguished by elegance and generality; to him is the merit of having been the first to avail himself of these methods in analysis, especially in the integral calculus."

The translation should therefore be well received in this country by those who are interested in the history of this branch of the subject. A full notice of the more general treatise by the same author must be deferred until the second volume is in our hands. The first volume contains a general introduction to the theory, with full solutions of a large number of typical examples, including, of course, those which may be called the "classical" problems. He then deals with the theory of errors, and the methods of least squares arising in connection therewith. Examples from different branches of science exhibit the application of this potent weapon in dealing with the manipulation of observations. Statistics and life assurance appear to be adequately dealt with, but we cannot speak of the treatment of the latter as a whole, as, according to German custom, the first volume ends abruptly in the middle of a paragraph. From what we have seen of Vol. I., the book is clearly of unusual merit. But to return. Interest in questions of "local probability" was first aroused by Buffon's famous "problem of the needle," and kindred problems. Until the English school took up the subject, aided and abetted by a few enthusiasts such as Lemoine, Jordan, and Lalanne, the thoughts of mathematicians were mainly directed to a few solitary and disconnected questions. Even Buffon, whose gifts were not of a mathematical order, cried out for more geometry in dealing with his needle, and his instinct was shared by Herr Czuber, as may be seen from the title of the book. In the days of Buffon it would have been impossible to prophesy the full scope which in the future was to be given to questions of this type. Yet so wide a field is covered by the applications of the theory, that it is of prime importance that we should feel some confidence as to its bases. One of the many difficulties surrounding the initial stages of investigation is very well put by Mr. Venn in his *Logic of Chance*: "It consists in choosing the class to which to refer an event, and therefore judging of the event and the improbability of foretelling it *after it has happened*, and then transferring the impressions we experience to a supposed contemplation of the event beforehand." And in illustration he gives the case of the man who exhibited a bull's eye on a small target as the result of his skill with an old fowling-piece at a distance of a hundred yards. The braggart had suppressed a fact which, after all, had something to do with the case. He had fired at the door, and had *then* placed a small chalk circle around the aperture made by the shot.

Again, let us take Chrystal's definition of probability. "If on taking a very large number N out of a series of cases in which an event A is in question, A happens on pN occasions, the probability of the event A is said to be p." Apply it to the question: I throw two dice, what is the chance that I throw at least one six? There are eleven unfavourable to thirty-six favourable cases, giving the probability $\frac{11}{36}$. But as far as the definition is concerned, why should we not treat it as follows: The number of combinations of the numbers on the dice is twenty-one, of which six are favourable, so that the probability is $\frac{6}{21}$?

Our definition is incomplete unless we add "assuming that the possible cases are equally probable." Which brings us to a *definition of the probable by the probable.* Or, again, the event may or may not happen—two possible cases, one favourable, hence the probability is $\frac{1}{2}$. Let us take another illustration from Bertrand. What is the chance that a random chord in a circle may be greater than the side of the inscribed equilateral triangle? If one extremity of the chord be known, the probability is unaffected thereby. The two sides of the equilateral triangle through this point make with the tangent at the point three angles of 60°. To be greater than the side of the triangle the chord must lie within the angle of the triangle. The chance of this is $\frac{1}{3}$. On the other hand, if we are given the direction of the chord, the probability is thereby unaffected. It is easily seen that the chance of this again is $\frac{1}{2}$. But we may also argue: to choose a random chord is to choose at random its middle point. The mid point of the chord must then be at a distance from the centre less than half the radius, *i.e.* within a circle one-fourth of the area of the original circle. The number of points within a circle four times less is four times less. Hence the chance is $\frac{1}{4}$.

Fortunately it is possible to find a portion of firm ground on which to take a stand in the midst of this quaking morass of haunting doubt. Were it not so, as M. Poincaré points out (*La Science et l'Hypothèse*, p. 217), we should involve in one common ruin the countless problems of science, in which probabilities play so important a part. It is not enough to say, "ce que je sais, ce n'est pas que telle chose est vraie, mais que le mieux pour moi est encore d'agir comme si elle était vraie. . ." We cannot, and we must not, condemn *en bloc*; discussion is as obligatory as it is inevitable. And in this connection it would be well for us to ponder once more over Kant's reflexions in the 4th Section of the *Antinomy of Pure Reason.* I am glad to be able to refer the reader to the extraordinarily interesting work of M. Poincaré. His concluding words are well worth bearing in mind: "There are certain points which seem well established. In any question of probability we must start with an hypothesis or a convention which will always contain something arbitrary. In the choice of this convention we can only be guided by the principle of sufficient reason. Unfortunately, this principle is entirely vague and elastic, and . . . assumes many different forms. The form under which we meet it most frequently is the belief in continuity, a belief which it would be difficult to justify by apodictic reasoning, but without which all science would be impossible. The problems to which the theory of probabilities can be profitably applied are those in which the result is independent of the initial hypothesis, with the sole proviso that the hypothesis satisfies the condition of continuity." The reader who is still dissatisfied must remember that, after all, the very name of the calculus of probabilities is in itself a paradox of the purest water. "La probabilité, opposée à la certitude, c'est ce qu'on ne sait pas, et comment peut-on calculer ce que l'on ne connaît pas?"

Histoire des Mathématiques dans l'Antiquité et le Moyen Age. By H. G. Zeuthen (translated from the Danish and German by J. Mascart). Pp. xvi., 296. 1902. (Gauthier-Villars.)

This translation of Professor Zeuthen's learned and popular work has the advantage of containing many additional notes and references from the pen of M. Tannery. There are various alterations in the sections on Trigonometry, due to the author's appreciation of Braunmühl's *Vorles. über Geschicht der Trigonometrie.* Speaking roughly, the volume before us deals mainly with Greek geometry. Euclid's elements from the *point de repère* from which are examined the logical forms so closely followed by the Greeks. The intrinsic nature of these forms apart from their significance to the minds of the early geometers. The student of Euclidean geometry will find the comments of the author of signal interest at the present time. Professor Zeuthen devotes some fifty pages to the investigations of the Arabs and Hindoos. The special aptitude of the Hindoos for numerical calculation is clearly expounded, as is also their extraordinary power of assimilating what they received from the West and applying the results to the production of original and independent results. This power the Arabs did not possess. None the less do we owe it to them that the cloisters of the Middle Ages by means of Arabian manuscripts were able by their translations into Latin to keep the lamp of science burning amid the darkness which was the natural

accompaniment of a general paralysis of intellectual activity. We hope that M. Mascart will be able to induce M. Gauthier-Villars to publish Professor Zeuthen's earlier work on the theory of Conics in antiquity.

Comptes rendus du deuxième congrès international des mathématiciens. (Paris, Aug. 6-12, 1900.) Edited by E. Duporcq. Pp. 450. 16fr. 1902. (Gauthier-Villars.)

This volume consists of the proceedings at the last Congress over which the illustrious Hermite presided. Space forbids our drawing attention to more than a few of the lectures and communications made to the Congress. Of the five formal lectures, one is by M. Mittag-Leffler on the correspondence between Weierstrass and Sophie Kowalevski. Incidentally the distinguished Swedish mathematician narrates the first words addressed to him on presenting himself as a student to attend Hermite's course at Paris: "Vous avez fait erreur, Monsieur; vous auriez dû suivre les cours de Weierstrass à Berlin. C'est notre maître à tous." This observation, three short years after *la funeste année*, 1870, shows how far the instincts of the Savant were wrought to finer issues than the purely local and patriotic sentiment of the day. M. Hilbert's lecture on the problems which are at present awaiting their solution by the hands of competent mathematicians has been translated into English by Dr. Mary Newson for the Bull. Math. Soc. (July, 1902). The problems are as follows: Cantor's problem of the cardinal number of the continuum; the compatibility of the axioms of arithmetic; the equality of the volumes of two tetrahedra of equal bases and equal altitudes; the problem of the straight line as the shortest distance between two points; Lie's concept of a continuous group of transformations without assuming that the functions defining the group are capable of differentiation; the treatment of the axioms of physics as we treat the axioms of mathematics, placing in the first rank probabilities and mechanics; the irrationality and transcendence of certain numbers; that the zero points of $\sum_{n=0}^{n=\infty}\frac{1}{n^s}$ have all the real part $\frac{1}{2}$, except the negative integral real zeroes; Riemann's prime number formula; Goldback's theorem, that every integer is expressible as the sum of two primes; is there an infinite number of pairs of primes differing by 2? is $ax+by+c=o$ soluble in prime numbers x and y, where a, b are integral and a prime to b? to apply the results obtained for the distribution of rational prime numbers to the theory of the distribution of ideal primes in a given number field k; and so on. M. Poincaré discoursed on the *rôle* of intuition and logic in mathematics, showing how, while intuition is often the source of discovery, it is logic which harmonises and consolidates the creations of intuition. This address is also published in a separate form (1fr.). Mr. H. Hancock's paper on Kronecker's Modular Systems defines congruences between algebraical integral numbers which are generalisations of the congruences of our elementary theory of numbers. That veteran American, Mr. Artemus Martin, gives new series for the calculation of logarithms, some of which converge with quite unusual rapidity. He also deals with expressions of the form $x^4+y^4+z^4+\ldots+u^4=k^4$. The Italians are following the lead of Professor Peano in their discussions on mathematical logic. Signor A. Padoa contributes two papers, one on a new irreducible system of postulates for algebra—seven in number, based on the (undefined) integer, its consecutive, and its image (symétrique)—and the other on a new system of definitions for Euclidean geometry—two in number, point and "may be superposed." Signor Veronese pleads for the reduction to a minimum of the postulates in teaching geometry. Professor A. Macfarlane applies space analysis to curvilinear co-ordinates; and Professor Stringham discusses orthogonal transformations in elliptic space. Signor Galdeano advocates the addition to the branches of Mathematics as taught at the Universities the principles of "Mathematical Criticism." It would consist of the study of the historical developments and the ties of kinship which link together the historical and the logical genesis of our knowledge. In a synthetic study of different branches "l'enchaînement des idées" must be fruitful of result. Signor A. Gallardo writes on the application of Mathematics to a complex subject, such as Biology. We must not forget to mention Mr. Fujisawa's engaging article on the Mathematics of the Old Japanese School. Few readers will be unable to discover in this *Compte Rendu* some paper or article in which they will be interested. It is a delightful collection.

Differential- und Integralrechnung. Vol. I. Differentialrechnung. By. W. F. Meyer. Pp. xviii., 395. 9m. (Göschen, Leipzig.) Vol. X. Sammlung Schubert.

This interesting volume is well worthy the careful perusal of teachers of the elements of the Calculus. It is especially remarkable for the prominence given at an early stage to the theory of errors. It is not until after some 70 pages of introductory matter that the author approaches the differential coefficient, the first chapter being devoted entirely to the fundamental conceptions. He begins with a theorem on the limits of a^n and a^m for $n=\infty$, applying them to the summation of a G.P. He proceeds at once to use these results in determining the areas of plane curves and the volumes of solids of revolution. In the same way other limits, such as $\frac{\sin\theta}{\theta}$, $\frac{\sin\theta}{\Pi\cos\frac{\theta}{2^n}}$, lead us to the tangents to conics. A section on the Binomial Theorem is followed by the determination of the equation of the tangents to a parabola of the m^{th} order and the elementary notions of integral functions. The second part is quite a monograph in itself on the development of series. Rolle's theorem and its applications, Taylor's and Maclaurin's Theorem, are treated at length; and the final sections form an adequate discussion on the convergence and divergence of series and of infinite products. The second volume will contain the applications of the differential calculus to curves and surfaces, and also a series of historical notes. To it we look forward with considerable interest. We can heartily commend this attractive little volume to the attention of those who are not satisfied with the ordinary introductions to elementary analysis.

PROBLEMS.

460. [R. 2. b. γ.] Find the centre of gravity of the part cut from a solid sphere by two diametral planes inclined at angle 2α to each other. ANON.

461. [M¹. 3. b.] MM' is a chord of a circle, centre O, parallel to a fixed diameter AB; MP is perpendicular to OM'. Find the areas of the locus of P and of the envelope of the line MP. What do they become when we replace the circle and its diameter by an ellipse and its major axis?

E. N. BARISIEN.

462. [K. 12. b. α.] Describe three circles, mutually tangent, to pass through three given points and touch a circle including the three points. A. B.

463. [L¹. 17. e.] Shew that the equation to a circumconic of the triangle ABC can be written in the form

$$\frac{a}{p\alpha}+\frac{b}{q\beta}+\frac{c}{r\gamma}=0,$$

where p, q, r are the lengths of the focal chords parallel to BC, CA, AB.

J. J. MILNE.

464. [K. 10. e.] A, B are two fixed points, P a variable point, on a circle; find the locus of the intersections of BP with the bisectors of the angle BAP.

V. RETALI.

465. [L¹. 6. c.] CP, CD are conjugate radii of an ellipse; PU, DV the chords of intersection of the ellipse with the circles of curvature at P, D; shew that CU, CV are conjugate. C. F. SANDBERG.

466. [K. 20. e.] In a triangle which has $\Sigma\cot A<2$, shew that the least angle $>\cot^{-1}\frac{4}{3}$ and the greatest $<90°$. If $\Sigma\cot A>2$, what is the greatest value of the least angle and the least value of the greatest? C. E. YOUNGMAN.

467. [L¹. 3. d.] Two tangents to a rectangular hyperbola meet on a fixed parabola having one asymptote of the hyperbola for axis and the other for the tangent at its vertex. Find the envelope of their chord of contact.

Durham, 1902.

468. [R. 4. c.] A parallelogram $ABCD$, formed of uniform rods of total weight W, smoothly jointed, is held in a vertical plane with AB vertical (A uppermost), and a light elastic string of natural length d and modulus W' connects the joints A, C. Find when in equilibrium the length of the string. Melbourne, 1901.

SOLUTIONS.

400. [K. 2. d; 5. d.] *A variable straight line meets two fixed straight lines Ox, Oy in A and B, so that the sum or difference of OA, OB is constant: find the envelope of the Euler circle of the triangle OAB, and the locus of its Lemoine point and the three associated points.*

Prove also that the circumcircle of OAB passes through another fixed point besides O, and that its Euler line also passes through a fixed point.

E. N. Barisien.

Solution by C. E. Youngman.

Let ω be the angle AOB. Draw the circle OAB (centre C) and the diameter DD' bisecting AB at M, so that OD is the internal bisector of the angle AOB. Then $DA=DB$, and angle $DAO=\pi-DBO$; therefore, if DA, DB be turned in the same direction through any angle θ to positions Dx, Dy, we have $Ax=By$ and $Ox+Oy=OA+OB$. Thus D is a fixed point when $OA+OB$ is constant; and similarly, D' is fixed when $OA \backsim OB$ is constant. Suppose the former: then, DA', DB' being perpendicular to OA, OB, M lies on $A'B'$ (the pedal line of D for OAB). Let fall AHE, BHF perpendicular to OB, OA; the circle MEF is the "Euler" circle of OAB (and surely "nine-point circle" is a better name, better known). Now M is the centre of the circle $ABEF$; therefore $ME=MF$, and angle $EMF=2EAF=\pi-2\omega$. Hence the triangle MEF has constant angles, and is inscribed to a fixed triangle $OA'B'$; therefore the circle MEF touches twice a fixed conic inscribed in $OA'B'$ (Neuberg, *Proc. Lond. Math. Soc.* xvi. 185). Also, since MEF is, in one position, the orthocentric triangle of $OA'B'$, the foci of the conic are the ortho- and circum-centres of $OA'B'$.

The Euler line CH divides OD in the ratio

$$OH : CD = 2CM : CA = 2\cos\omega : 1 = \text{const.};$$

i.e. it cuts OD at a fixed point P.

Let $KK_1K_2K_3$ be the Lemoine point and its associates. K_1 is the pole of AB for the circle OAB; therefore $CM=CA\cos\omega=CK_1\cos^2\omega$, and $CM:CK_1=\text{const.}$ Now C and M move on lines perpendicular to OD; therefore K_1 on another. K_2 is the pole of a fixed line OB for a set of circles OD, that is, of conics passing through four fixed points—O, D, and the two at infinity; therefore K_2 describes a conic—a hyperbola, for K_2 goes to infinity when C is on OB or at infinity. K_3, of course, describes an equal hyperbola. As for K, it is the mean centre of O, A, B for multiples AB^2, BO^2, OA^2, or $a^2+b^2-2ab\cos\omega$, b^2, a^2. Hence its coordinates are given by

$$2x(a^2+b^2-ab\cos\omega)=ab^2, \quad \text{and} \quad 2y(a^2+b^2-ab\cos\omega)=a^2b.$$

From these and $a+b=c$ eliminate a and b; the result is

$$2(x+y)(x^2+y^2-xy\cos\omega)=cxy;$$

so that the locus of K is a cubic which has a node at O, and an asymptote parallel to OD'.

413. [K.] *Along three straight lines meeting in O at angles α, β, γ, lengths x, y, z are measured: prove that the three points so obtained lie on a straight line if $\Sigma x^{-1}\sin\alpha=0$, and on a circle through O if $\Sigma x\sin\alpha=0$.*

R. W. H. T. HUDSON.

Solution by C. V. DARELL, E. FENWICK, *and others.*

Let P, Q, R be the three points.

Then since P, Q, R are *collinear*, we have $\Sigma\triangle QOR=0$;

$$\therefore\ \Sigma yz\sin\alpha=0.$$

$$\therefore\ \Sigma x^{-1}\sin\alpha=0.$$

If P, Q, R lie on a circle through O, invert with respect to O, K being the constant of inversion, then the points whose distances from O measured along the three fixed lines are $\frac{K^2}{x}$, $\frac{K^2}{y}$, $\frac{K^2}{z}$ must be *collinear.*

Therefore, using previous result,

$$\Sigma x\sin\alpha=0.$$

BOOKS, ETC., RECEIVED.

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ERRATA.

p. 224, for Note 109 read 117.
p. 225, for Note 110 read 118.
p. 225, to Note 110 attach at end—E. L. BULMER.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO.